Invariant sets method for state-feedback control design

Anamaria Luca, Pedro Rodriguez-Ayerbe and Didier Dumur

***Abstract*— This paper presents a invariant sets approach for the state-feedback control design. Considering a linear discrete time-varying system with polytopic uncertainty, affected by state disturbances and input/output constraints, the proposed methodology gives the best control law in terms of finding the maximal invariant ellipsoid where stability and constraints satisfaction are assured. In the second part of the paper a robust performance criterion is considered, in order to ajust the robustness and the performance. His effects on the invariant maximal ellipsoid and the obtained performance are discussed.**

Key–Words: Constraints, Ellipsoids, Invariants sets, Linear Matrix Inequalities, Lyapunov functions, S–procedure.

I. INTRODUCTION

In classical robust design, a controller satisfying robust stability and performance is searched for a considered uncertain system. Usually, classical techniques do not consider the constraints of the system, but in recent years, several control techniques using invariant sets have been studied in order to accomplish that. This paper proposes a method for state feedback design using invariant sets techniques. The objective is to synthesize a feedback gain that guarantees the robust stability and some performance in the biggest region of the space satisfying the constraints. Are considered constraints on the input, as for example the ones given by the saturation of the actuators, and on the output, as for example for guaranteeing the current level in a self, in order to not saturate the magnetic core, or on the output level in a hydraulic system.

A *positively invariant* set can be on short defined like a subset of the space state with the property that, if it contains the system state at some time, then it will contain it also in the future. This means that once the state is in the set it will never exit. A set is said to be *invariant* if the inclusion of the state at some times implies the inclusion in both the future and the past [1]. In the presence of disturbances, if the invariance is preserved, the term of *robust invariance* is used.

Since the existence of an invariant set is equivalent with the existence of a Lyapunov function, the invariant set theory provides a suitable theoretical framework to deal with stability problem. In the presence of disturbances or uncertainties the notion of input-to-state stability (ISS) is used since is not possible to guarantee that the origin is asymptotically stable. ISS implies that the origin is an asymptotically stable point for the nominal system $x(k+1) = f(x(k), v = 0)$ with region of attraction $\mathfrak{X} \subseteq \mathbb{R}^n$, and also that all state trajectories are bounded for all bounded disturbance sequences. Furthermore every trajectory $\Phi(x, k, v(\cdot)) \to 0$ if $v(k) \to 0$ as $k \to \infty$ where $v(\cdot)$ represents the noise [2].

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In the past years, set invariance theory was intensely studied given the benefits that it has to offer. A broad lecture about this family of sets is well outlined in [1]. Once the invariance theory developed, methods for obtaining invariant sets were searched. In [3], [4] are considered invariant ellipsoidal sets that are determined using LMI techniques.

The paper is organized as follows. The class of systems that is to be considered is presented in Section 2 along with some basic informations about LMIs. In Section 3 the main results are presented consisting in finding a maximal invariant ellipsoid and the corresponding state-feedback law. In order to state the results a numerical example is presented in Section 4. In Section 5 some concluding remarks are stated.

The notations are standard.

II. PRELIMINARIES

A. System model

Consider the following discrete linear time-varying (LTV) system:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + B_\omega \omega(k) \\ y(k) &= Cx(k) \\ \begin{bmatrix} A(k) & B(k) \end{bmatrix} &\in \Omega, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the output, $\omega \in \mathbb{R}^{n_\omega}$ the state noise and Ω is a polytope

$$\Omega = Co\{[A_1\ B_1],\ [A_2\ B_2], ..., [A_L\ B_L]\}, \tag{2}$$

with Co devoting to the convex hull. This means that if $[A\ B] \in \Omega$ then for some $\lambda_i \geq 0$, $\sum_{i=1}^{L} \lambda_i = 1$ we have $[A\ B] = \sum_{i=1}^{L} \lambda_i [A_i\ B_i]$, where A_i, B_i, $i = 1...L$ are vertices of the uncertain polytope Ω. $L = 1$ corresponds to the nominal LTI (linear time invariant) system description.

The control law has the form:

$$u(k) = Fx(k) \tag{3}$$

where $F \in \mathbb{R}^{m \times n_x}$ is a fixed feedback gain matrix such that $A + BF$ is strictly stable.

B. System constraints

We consider Euclidean norm bounds on the control input:

$$\| u \|_2 \leq u_{max} \tag{4}$$

Similarly, for the output, we consider the Euclidean norm constraint:

$$|| y ||_2 \leq y_{max} \quad (5)$$

Also, the disturbance vector ω is bounded:

$$\omega^T \omega \leq 1 \quad (6)$$

Remark 1: Constraints on the input are considered hard constraints since they are usually limitations on process equipment. Output constraints are typically called soft constrains since they are often performance goals and so they can be softened.

C. Linear matrix inequalities

A linear matrix inequality or LMI is a matrix inequality of the form

$$F(x) = F_0 + \sum x_i F_i \succ 0,$$

where $x_1, x_2, ..., x_n$ are the variables, $F_i = F_i^T \in \mathbb{R}^{n \times n}$ are given, and $F(x) \succ 0$ means that $F(x)$ is positive-definite [5].

Schur complements states that for a $Q(x) = Q(x)^T$, $R(x) = R(x)^T$ and $S(x)$ depend affinely on x, the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} \succ 0$$

is equivalent to the matrix inequalities

$$R(x) \succ 0, Q(x) - S(x)R(x)^{-1}S(x)^T \succ 0$$

or, equivalently,

$$Q(x) \succ 0, R(x) - S(x)^T Q(x)^{-1} S(x) \succ 0 \text{ [5].}$$

III. MAIN RESULTS

Over the years several families of invariant sets have been considered in literature, a very popular class of invariant sets is that of ellipsoidal sets or ellipsoids. An ellipsoidal set can be defined as follows:

$$E = \{x \mid x^T G^{-1} x \leq 1\} \quad (7)$$

where $G^{-1} = P \in \mathbb{R}^{(n_x) \times (n_x)}$ is a symmetric positive-definite matrix. One of the reasons for choosing ellipsoidal invariant sets is their connection with powerful tools such as Lyapunov function or LMI techniques, another reason for choosing ellipsoids despite polytopic invariant sets is given by the fact that the last ones do not lend themselves to analysis.

The aim of this paper is to find the maximal invariant set and the stabilizing control law that provides this set for a polytopic system, affected by disturbances and input and output constraints. Before enunciating the theorem that gives the maximal ellipsoid and the stabilizing state-feedback law, some notions about S-procedure have to be given since it has a major importance in obtaining the results.

S-procedure Let $F_0 = F_0^T$, $F_1 = F_1^T \in \mathbb{R}^{n \times n}$. For all z satisfying $z^T F_1 z \geq 0$ implies $z^T F_0 z \geq 0$ if exists an $\alpha \in \mathbb{R}$, $\alpha \geq 0$ with $F_0 \succeq \alpha F_1$ [6].

Theorem 1: Consider discrete linear time-varying system (1) with the control law given by (3). The offline maximization of E subject to input constraint (4), output constraints (5) and noise presence ω (6) is performed by solving the SDP (semi-definite programming):

$$\max_{G,Y} \ \log \ \det G \quad (8)$$

subject to:

$$\begin{bmatrix} G & 0 & \alpha G & GA_i^T + Y^T B_i^T \\ 0 & \alpha I & 0 & B_\omega^T \\ \alpha G & 0 & \alpha G & 0 \\ A_i G + B_i Y & B_\omega & 0 & G \end{bmatrix} \succeq 0 \quad (9)$$

$$\begin{bmatrix} G & Y^T \\ Y & u_{max}^2 I \end{bmatrix} \succeq 0 \quad (10)$$

and

$$\begin{bmatrix} G & (A_i G + B_i Y)^T C^T \\ C(A_i G + B_i Y) & y_{max}^2 I - C B_\omega B_\omega^T C^T \end{bmatrix} \succeq 0 \quad (11)$$
$$i = 1, 2, ..., L$$

The stabilizing feedback gain that maximizes the invariant ellipsoid is $F = YG^{-1}$.

Proof. Let the Lyapunov function $V = x^T P x$, $P = P^T = G^{-1} \succ 0$. Invariance implies that $-\Delta V = V(k) - V(k+1) \geq 0$. This leads to:

$$\begin{bmatrix} x \\ \omega \end{bmatrix}^T \begin{bmatrix} P - (A_i + B_i F)^T P (A_i + B_i F) & \star \\ -B_\omega^T P (A_i + B_i F) & -B_\omega^T P B_\omega \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \geq 0 \quad (12)$$

where $\star$ represents the transpose element.

Also $x^T P x \geq 1$ and $\omega^T \omega \leq 1$ can be written like:

$$\begin{bmatrix} x \\ \omega \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \geq 0 \quad (13)$$

From S-procedure we have that (13) implies (12) if exist $\alpha \geq 0$ such that:

$$\begin{bmatrix} P - (A_i + B_i F)^T P (A + BF) & \star \\ -B_\omega^T P (A_i + B_i F) & -B_\omega^T P B_\omega \end{bmatrix} \succeq \alpha \begin{bmatrix} P & 0 \\ 0 & -I \end{bmatrix}$$

This can be written like:

$$\begin{bmatrix} P & 0 \\ 0 & \alpha I \end{bmatrix} - \begin{bmatrix} P & 0 \\ P(A_i + B_i F) & P B_\omega \end{bmatrix}^T \begin{bmatrix} \alpha P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} P & 0 \\ P(A_i + B_i F) & P B_\omega \end{bmatrix} \succeq 0.$$

By applying Schur complements and considering $\alpha > 0$ (which is the case for the considered system) one obtains:

$$\begin{bmatrix} P & 0 & P & (A_i + B_i F)^T P \\ 0 & \alpha I & 0 & B_\omega^T P \\ P & 0 & \frac{1}{\alpha} P & 0 \\ P(A_i + B_i F) & P B_\omega & 0 & P \end{bmatrix} \succeq 0 \quad (14)$$

By pre- and post-multiplying this inequality with $\mathrm{diag}(G, I, \alpha G, G)$ and making the substitution $Y = FG$ the LMI (9) yields.

For proving the LMI corresponding to input constraints we have:

$$\begin{aligned} || u ||_2^2 &= || Fx ||_2^2 \leq || F P^{-1/2} ||_2^2 || P^{1/2} x ||_2^2 = \\ &= \lambda_{max}(F P^{-1} F^T)(x^T P x) \leq \lambda_{max}(F P^{-1} F^T) \end{aligned}$$

where λ_{max} is the maximal eigenvalue.

Now, using Schur we have that $\|u\|_2 \leq u_{max}$ if:

$$\begin{bmatrix} P & F^T \\ F & u_{max}^2 I \end{bmatrix} \succeq 0$$

Pre- and post-multiplying this inequality with $\mathrm{diag}(G,I)$ and using again Schur gives LMI (10).

Now for the output constraints one gets:

$$\begin{aligned} \|y\|_2^2 &= \|C(A_i+B_iF)x(k)+CB_\omega\omega(k)\|_2^2 \leq \\ &\leq \|C(A_i+B_iF)x(k)\|_2^2+\|CB_\omega\omega(k)\|_2^2 \leq \\ &\leq \lambda_{max}[C(A_i+B_iF)P^{-1}(A_i+B_iF)^TC^T](x^TPx)+ \\ &+ \lambda_{max}[CB_\omega B_\omega^TC^T](\omega^T\omega) \leq \lambda_{max}[CB_\omega B_\omega^TC^T]+ \\ &+ \lambda_{max}[C(A_i+B_iF)P^{-1}(A_i+B_iF)^TC^T] \end{aligned}$$

With Schur theorem we obtain:

$$\begin{bmatrix} P & (A_i+B_iF)^TC^T \\ C(A_i+B_iF) & y_{max}^2 I - CB_\omega B_\omega^TC^T \end{bmatrix} \succeq 0$$

By congruence with $\mathrm{diag}(G,I)$, LMI (11) yields. For completing the proof we must say that the feedback F is obtained by $F = YG^{-1}$. ∎

Remark 2: S-procedure introduces a new variable α. The presence of α render the inequality (9) BMI (bilinear matrix inequality). Because α is a scalar, an α_{optim} can be found by executing a simple loop. Another simple way is to use the PENBMI solver [7] (or other solvers) in MatLab environment which proved to work successfully.

Remark 3: Due to the fact that the ellipsoid volume is inversely proportional with the eigenvalues product (the determinant), finding the maximal ellipsoid is done by solving the problem $max\ det(G)$. For rendering the problem convex, the "logarithm" operator is used. Because the MatLab tools we use are build to find the minimum of a convex problem, our optimization criteria becomes $min\ -log\ det(G)$ [8].

The previous theorem gives the necessary and sufficient conditions for obtaining stability. In order to achieve robust performance we impose a upper bound to ΔV that guarantees a certain decreasing for the Lyapunov function:

$$V(k+1)-V(k) \leq -\frac{1}{\gamma}(x(k)^TQx(k)+u(k)^TRu(k)) \quad (15)$$

Theorem 2: Consider discrete linear time-varying system (1) with the control law given by (3). The offline maximization of E subject to noise presence ω (6), input constraint (4), output constraints (5) and robust performance constraints (15) is performed by solving the SDP (semi-definite programming):

$$\max_{G,Y}\ \log\ \det G \quad (16)$$

subject to: (10), (11) and

$$\begin{bmatrix} G & 0 & \alpha G & GA_i^T+Y^TB_i^T & GQ^{\frac{1}{2}} & YR^{\frac{1}{2}} \\ 0 & \alpha I & 0 & B_\omega^T & 0 & 0 \\ \alpha G & 0 & \alpha G & 0 & 0 & 0 \\ A_iG+B_iY & B_\omega & 0 & G & 0 & 0 \\ Q^{\frac{1}{2}}G & 0 & 0 & 0 & \gamma I & 0 \\ R^{\frac{1}{2}}Y & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \quad (17)$$

i = 1,2,...,L

The stabilizing feedback gain that maximizes the invariant ellipsoid is $F = YG^{-1}$.

Proof. Considering (13) and S-procedure, equation (15) is equivalent with:

$$\begin{bmatrix} P-\frac{1}{\gamma}(Q+F^TRF) & 0 \\ 0 & \alpha I \end{bmatrix} - \begin{bmatrix} P & 0 \\ P(A_i+B_iF) & PB_\omega \end{bmatrix}^T \cdot \\ \cdot \begin{bmatrix} \alpha P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \begin{bmatrix} P & 0 \\ P(A_i+B_iF) & PB_\omega \end{bmatrix} \succeq 0.$$

For $\alpha > 0$, by applying Schur complements one gets:

$$\begin{bmatrix} P & 0 & P & (A_i+B_iF)^TP \\ 0 & \alpha I & 0 & B_\omega^TP \\ P & 0 & \frac{1}{\alpha}P & 0 \\ P(A_i+B_iF) & PB_\omega & 0 & P \end{bmatrix} - \\ \begin{bmatrix} Q^{\frac{1}{2}} & F^TR^{\frac{1}{2}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma}I & 0 \\ 0 & \frac{1}{\gamma}I \end{bmatrix} \begin{bmatrix} Q^{\frac{1}{2}} & 0 & 0 & 0 \\ R^{\frac{1}{2}}F & 0 & 0 & 0 \end{bmatrix} \succeq 0$$

By applying again Schur complement, pre-and post-multiplying the resulting inequality with $\mathrm{diag}(G,I,\alpha G,G,I,I)$ and making the substitution $Y = FG$, LMI (16) yields. ∎

As expected, the robust performance criterion is a compromise between the maximal ellipsoid volume and the system reaction speed. By imposing this criterion, the largest invariant ellipsoid will lose in volume but the system will gain in speed response.

IV. NUMERICAL EXEMPLE

Consider the polytopic system (given in [5]) affected by disturbances and with input and output constraints. The system is in form (1), Ω being defined by (2) with:

$$A_1 = \begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.0591 & 0.2641 \\ 1.7971 & 0.8717 \end{bmatrix},$$
$$B = \begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } B_\omega = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}.$$

We impose the control constraint $\|u\|_2 \leq u_{max} = 1V$ and the output constraint $\|y\|_2 \leq y_{max} = 1V$.

By applying theorem 1, the optimum α for which we have the largest ellipsoidal invariant set is $\alpha_{opt} = 0.00656$. For this α the maximal ellipsoid has the volume $V_{max} = 42.9184$ and the control law that gives this volume is: $F = \begin{bmatrix} 0.3269 & 0.2514 \end{bmatrix}$.

In Fig. 1 we have the representation of the maximal invariant ellipsoid. The diagonal band represents the state space where the constraints are satisfied. It can be seen that the ellipsoid is inside this area, assuring the constraints satisfaction. For proving invariance we considered some different initial points for the state and plotted their trajectories. In the complementary figure the ISS property is pointed out: noise presence and uncertainty do not allow the state to asymptotically converge to 0 but instead, the state converge to an attraction region who in fact is the minimal invariant ellipsoid (the smallest ellipsoid where invariance

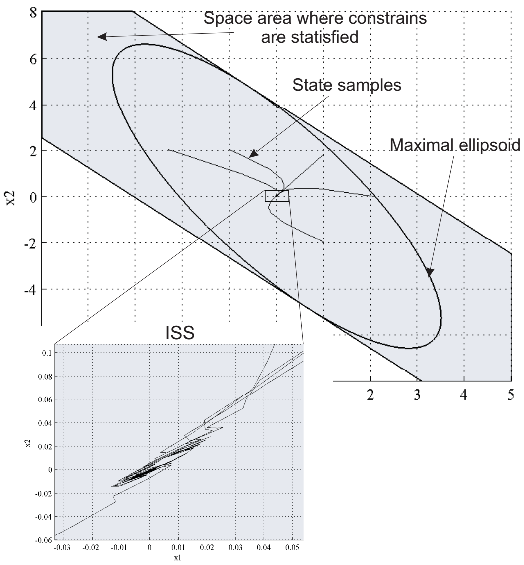

Fig. 1. Maximal invariant ellipsoidal set.

and constraints are satisfied under noise and uncertainty presence).

Considering now the robust performance criterion given by (15) with $\gamma = 30, Q = 1.5$ and $R = 1$, the ellipsoid drawn in Fig. 2 yields.

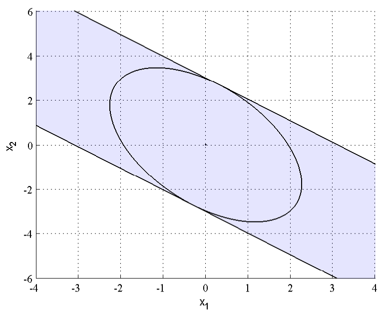

Fig. 2. Maximal invariant ellipsoidal set satisfying robust performance conditions.

In this case we obtain $\alpha_{opt} = 0.00281$ and the maximal volume $V_{max} = 21.1021$. The control law that gives this volume is: $F = \begin{bmatrix} 0.3223 & 0.3335 \end{bmatrix}$.

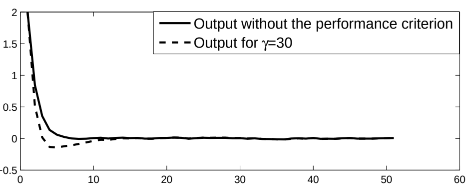

Fig. 3. System output considering or not the robust performance criterion.

The robust performance criterion reduced the invariant ellipsoid size but improves the system comportment in terms of response speed. In Fig. 3 we plotted the output for the case with performance criterion (solid line) and without the criterion (dashed line). It can be seen that the robust performance criterion presence has increased the speed with which the output converges to the reference.

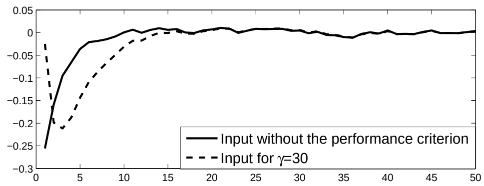

Fig. 4. System input considering or not the robust performance criterion.

The same observation can be made for the input (Fig. 4): the input trajectory in the presence of the robust performance criterion (the solid line) reaches the attraction area faster than in the absence of the criterion (the dashed line).

For obtaining these results MatLab environment was used. The optimization were solved using the software Yalmip [9] with the Sedumi solver [10] in MatLab environment.

V. CONCLUSION

This paper presents a simple and clear approach for determining the largest invariant ellipsoid and the state-feedback gain that assures the maximality of the invariant set. The maximal ellipsoid provides the biggest x-subspace region where, for a uncertain system affected by bounded disturbances and constraints, we can assure invariance, constraint satisfaction and obviously ISS. In order to accomplish robust performance, an upper bound is imposed to the Lyapunov function. This upper limit has as result a compromise between the maximal invariant set volume and the convergence speed of the considered system. The problems presented here are solved efficiently by LMI solvers with a reduced computational burden.

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