Nonuniform Sampled Signal Reconstruction in the L_1 Norm

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***Abstract* — This paper considers the problem of reconstructing a bandlimited signal from a finite number of its non-uniformly distributed samples. We evaluate a new algorithm for this purpose and compare it to an existing algorithm. We analyze the advantages and disadvantages of both algorithms. Reconstruction efficiency and accuracy of approximation procedures differ depending of the number of missed samples in nonuniformly sampled signals. Moreover the number of iterations during reconstruction phases affects approximation error.**

***Keywords* — Approximation, nonuniform sampling, reconstruction.**

I. INTRODUCTION

DIGITAL signal processing samples continous-time signals in order to obtain discrete-time representations. Shannon's sampling theorem for band-limited signals states that the signal is uniquely determined by its values at an infinite set of sample points spaced $½(f_m)^{-1}$ seconds apart, where f_m is the maximal frequency in the signal's spectrum.

In real-world measurement systems the data may suffer from several problems, including data dropouts, an irregularly spaced sampling grid and time delayed sampling, which produce a subset of non-uniformly distributed samples.

It is therefore important to understand how to reconstruct signals from a smaller number of non-uniformly distributed measurements than the number required by the Shannon's sampling theorem. It requires developing new efficient reconstruction algorithms and implementing them numerically.

In this paper we propose a basis fitting algorithm, based on multiresolution and wavelet coeficient estimation, as an extension of the Multiresolutional Basis Fitting Reconstruction (MBFR) method proposed in [1], and the Multiresolutionally-based Conjugate Gradients (MCG) algorithm developed in [2]. The principal difference is that the MCG algorithm applies L_2 – norm approximation while we apply L_1 – norm approximation as in [3].

Section II introduces the idea of the basis fitting method and describes the improved algorithm for solving overdetermined systems of linear equations in the L_1 norm. We show how to apply the algorithm on suitable chosen examples in section III.

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II. THEORETICAL BACKGROUND

A. Discrete Wavelet Transform

The DWT (Discrete Wavelet Transform) decomposes a signal into a set of orthogonal components describing the signal variation across scales [4]. In analogy with other function expansions, a function f may be written, for each discrete coordinate t, as sum of a wavelet expansion up to certain scale J plus a residual term, that is:

$$f(t) = \sum_{j=1}^{J} \sum_{k=1}^{2^{-j}M} d_{jk}\psi_{jk}(t) + \sum_{k=1}^{2^{-J}M} c_{Jk}\phi_{Jk}(t) \qquad (1)$$

Equation (1) states that at any given position t, the value of $f(t)$ is given by a sum over all dilations $j = 1,\dots, J$, and over all translations $k = 1,\dots, 2^{-j}M$, of the wavelet function $2^{-j/2}\psi(2^{-j}t-k)$ multiplied by its estimated coefficients d_{jk} plus a residual that corresponds to a coarse approximation of $f(t)$ at resolution J. This term is given by the scaling function $\phi_{Jk}(t)$ multiplied by its coefficients c_{Jk}.

The estimation of d_{jk} and c_{Jk} is carried out through an iterative decomposition algorithm, which uses two complementary filters H_0 (low-pass) and H_1 (high-pass). Since the wavelet base is orthogonal, H_0 satisfies the quadrature mirror filter conditions [4].

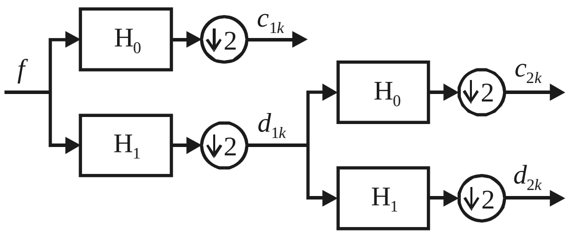

Fig. 1. Discrete wavelet transform tree.

The algorithm starts by passing the data through both filters H_0 and H_1 (Fig. 1.), and then decimates their output by half. The high-pass filtered and decimated data are the wavelet coefficients d_{jk} for the finest resolution. The low-pass filtered and decimated data are the coefficients c_{Jk} of the scaling function. By reapplying H_0 and H_1 to the residuals c_{Jk}, one obtains the wavelet and scaling coefficients for the coarser resolution, and so on.

The inverse transform can be obtained simply by reversing the previous sequence and by use of synthesis filters F_0 and F_1, which are the reflection of H_0 and H_1. The inverse DWT is illustrated in Fig. 2.

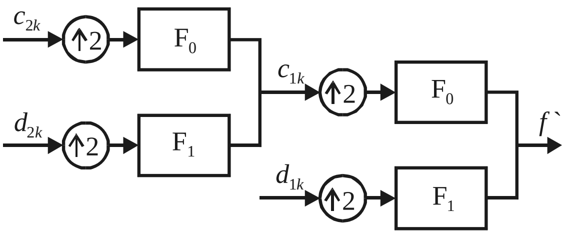

Fig. 2. Inverse wavelet transform tree.

B. *Description of the problem*

Suppose we want N uniformly distributed samples of a discrete signal $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]^T$, but we have only a subset of non-uniformly spaced samples $\mathbf{y} = [y_0, y_1, \ldots, y_{P-1}]^T$, where $P \le N$. Our goal is to obtain signal $\mathbf{x}' = [x'_0, x'_1, \ldots, x'_{N-1}]^T$, which is the best approximation of original uniformly sampled signal $\mathbf{x}$.

We assume that the non-uniformly sampled signal is under-sampled with respect to the Nyquist frequency of the complete uniformly sampled signal.

Suppose that the non-uniformly sampled signal $\mathbf{y}$ has been obtained through sub-sampling of the signal $\mathbf{x}$, as expressed by the following equation:

$$\mathbf{y} = \mathbf{H}_s \mathbf{x} \tag{2}$$

where $\mathbf{H}_s$ is a sub-sampling matrix, with entries 0 and 1.

In the reconstruction procedure, we can start from any resolution level J, but without loss of generality we explain the procedure using the synthesis bank depicted on Fig. 2. Since we do not have all of the samples on the desired grid, we cannot compute the wavelet coefficients of the signal at the highest resolution level. We can however, approximate the signal by its low frequency components, temporarily ignoring the high-frequency terms. These assumptions produce the predetermined linear system:

$$\mathbf{H}_s \mathbf{F}_0 (\uparrow 2) \mathbf{F}_0 (\uparrow 2) \mathbf{c}_2 = \mathbf{y} \tag{3}$$

We are looking for a best approximation to this system. The scaling function coefficients, represented by vector $\mathbf{c}_2$, are estimated by applying an algorithm for solution of overdetermined linear system in the L_1 norm [3]. The estimates $\mathbf{c}'_2$ yield a low frequency estimate $\mathbf{x}'$ denoted by $\mathbf{x}'_a$:

$$\mathbf{x}'_a = \mathbf{F}_0 (\uparrow 2) \mathbf{F}_0 (\uparrow 2) \mathbf{c}'_2 \tag{4}$$

In the next step, we calculate the error signal at each available sample:

$$\mathbf{e}_2 = \mathbf{y} - \mathbf{H}_s \mathbf{x}_a \tag{5}$$

Similarly as above, we can estimate the detail coefficients of the second level decomposition by solving the predetermined linear system:

$$\mathbf{H}_s \mathbf{F}_0 (\uparrow 2) \mathbf{F}_1 (\uparrow 2) \mathbf{d}_2 = \mathbf{e}_2 \tag{6}$$

From available estimates, $\mathbf{d}'_2$, we can calculate the difference signal at the finer level:

$$\mathbf{e}'_1 = \mathbf{e}'_2 - \mathbf{H}_s \mathbf{F}_0 (\uparrow 2) \mathbf{F}_1 (\uparrow 2) \mathbf{d}'_2 \tag{7}$$

and the following system is generated:

$$\mathbf{H}_s \mathbf{F}_1 (\uparrow 2) \mathbf{d}_1 = \mathbf{e}_1 \tag{8}$$

By solving this predetermined system we estimate the detail coefficients $\mathbf{d}'_1$. At the end, the reconstructed signal can be calculated using the following expression:

$$\begin{aligned} \mathbf{x}' = {} & \mathbf{F}_0 (\uparrow 2) \mathbf{F}_0 (\uparrow 2) \mathbf{c}'_2 + \mathbf{F}_0 (\uparrow 2) \mathbf{F}_1 (\uparrow 2) \mathbf{d}'_2 + \\ & + \mathbf{F}_1 (\uparrow 2) \mathbf{d}'_1 \end{aligned} \tag{9}$$

Since the matrices that appear in (9), (12) and (14) contain a significant number of zero-valued elements, by applying sparse matrix techniques, the computational time and the number of floating point operations are reduced significantly.

C. *Algorithm for Discrete L_1 Linear Approximation*

The general L_1 - norm linear approximation problem can be described as follows. Let $f(x)$ be a given real-valued function defined on a discrete subset: $X = \{x_1, x_2, \ldots, x_m\}$ of euclidian space E^N. Given $n(\le m)$ real-valued functions $\varphi_j(x)$ defined on X, we form a linear approximation function

$$L(A, x) = \sum_{j=1}^{n} a_j \varphi_j(x) \tag{10}$$

for any set $A = \{a_1, a_2, \ldots\ldots, a_n\}$ of real numbers. The L_1 – norm approximation problem is to determine a best approximation $L(A^*, x)$ which minimazes

$$\sum_{i=1}^{m} |f(x_i) - L(A, x_i)| \tag{11}$$

For the L_1 – norm approximation problem let us write

$$\varphi_{ji} \equiv \varphi_j(x_i),\ f_i = f(x_i) \tag{12}$$

and define non negative variables u_i, v_i, b_j, c_j by putting

$$f_i - \sum_{j=1}^{n} a_j \varphi_{ji} = u_i - v_i \quad i = 1, 2, \ldots, m \qquad (13)$$

And $a_j = b_j - c_j$ for $j = 1, 2, \ldots, n.$ Then a best L_1 approximation coresponds to an optimal solution to the linear programming problem:

$$\textit{Minimize} \ \sum_{i=1}^{m} u_i + v_i$$

$$\textit{subject to} \ f_i = \sum_{j=1}^{n} (b_j - c_j) \varphi_{ji} + u_i - v_i \qquad (14)$$

$$i = 1, 2, \ldots, m,$$

$$\textit{and} \quad b_j, c_j, u_i, v_i \geq 0$$

The dual of (14) is stated most conviniently as the bounded-variable linear programming problem:

$$\textit{Maximize} \ \sum_{i=1}^{m} (w_i f_i + f_i)$$

$$\textit{subject to} \sum_{i=1}^{m} w_i \varphi_{ji} = \sum_{i=1}^{m} \varphi_{ji} \qquad (15)$$

$$j = 1, 2, \ldots, n,$$

$$\textit{and} \quad 0 \leq w_i \leq 2$$

III. Experimental results

This section contains experimental results obtained with the proposed L_1 - norm approximation algorithm using a second level wavelet decomposition tree. We compare this to results obtained by the MCG algorithm, which use L_2 - norm approximation. Experiments are performed on non-uniformly sampled signals with a different number of missed samples. All simulations are carried out under the same condition for both algorithms and the results are presented in graphical and numerical form.

The next figure shows the signal which is non-uniformly sampled with 45% missed samples.

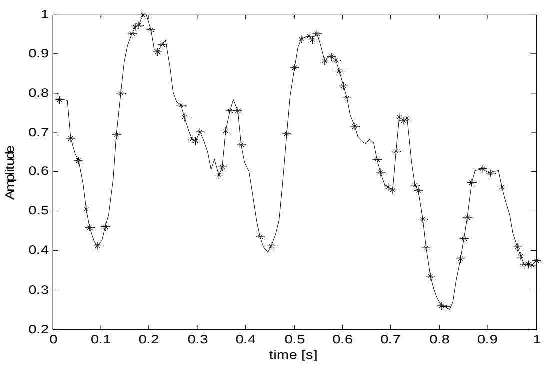

Fig. 3. Original signal (solid line) and non-uniformly sampled subset (*).

Reconstruction procedure uses a two-stage filter bank as depicted in Fig. 2. Approximation algorithm is carried out in three phases. After first level a scaling function coefficients are estimated. Second and third phase of approximation procedure make correction by adding details coefficients of the wavelet transform.

Fig. 4. shows the estimations of approximation and details coefficients of the wavelet transform.

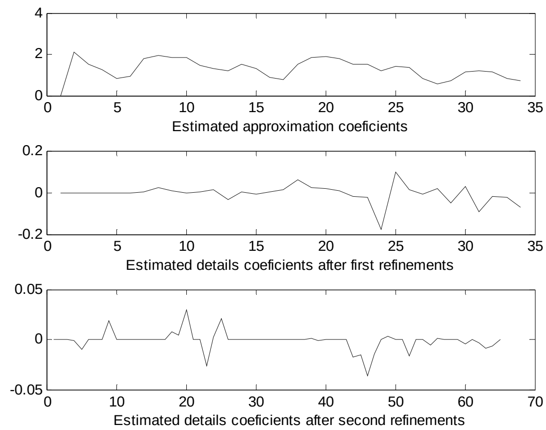

Fig. 4. Estimated scaling function and wavelets coefficients during signal reconstruction procedure with db2 basis and applying L_1 approximation.

Approximation function convergence during three-stage refinements procedure is presented at Fig. 5.

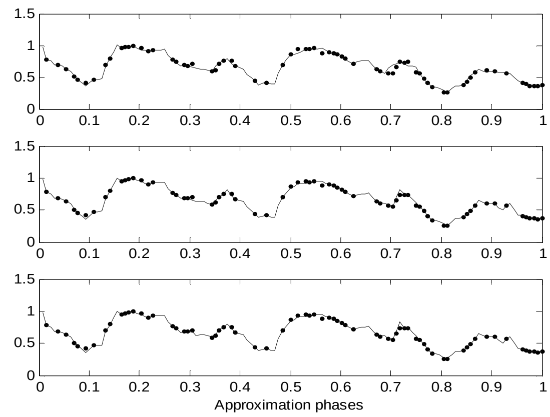

Fig. 5. Signal reconstruction phases.

The next two figures present results of reconstruction with two different approximation norm criteria.

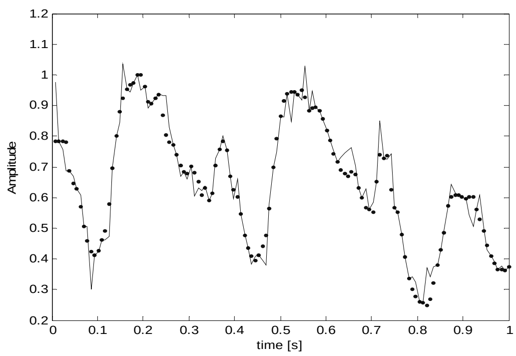

Fig. 6. Reconstructed signal using db2 basis applying L_1 approximation.

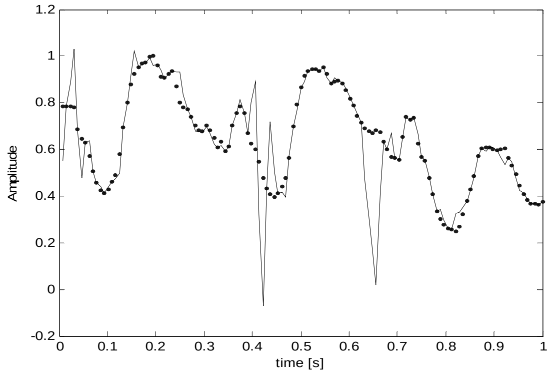

Fig. 7. Reconstructed signal using db2 basis applying MCG, L_2 approximation

TABLE 1. NUMERICAL RESULTS OBTAINED IN THE SIMULATION PROCESS FOR DIFFERENT RECONSTRUCTION ALGORITHMS

Algorithm	***Elapsed time***	$\|\|x - x_1\|\|$
L_1, db2	0,297 sec.	0,5242
MCG Db2, 2 level	0,296 sec.	1,3312

Fig. 6. shows an original uniformly sampled signal of length 128 and its non-uniformly distributed subset that comprises 70% of the signal. The next two figures show reconstructed signals obtained by the L_1 approximation and MCG algorithm respectively.

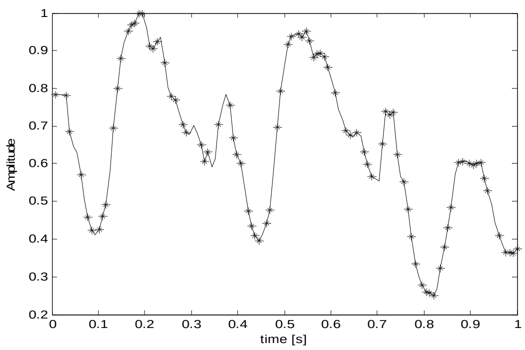

Fig. 8. Original signal (solid line) and non-uniformly sampled subset (*).

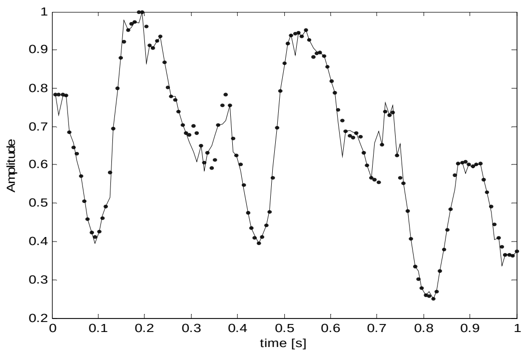

Fig. 9. Reconstructed signal using L_1 - norm algorithm (solid line) and original uniformly sampled signal (dots).

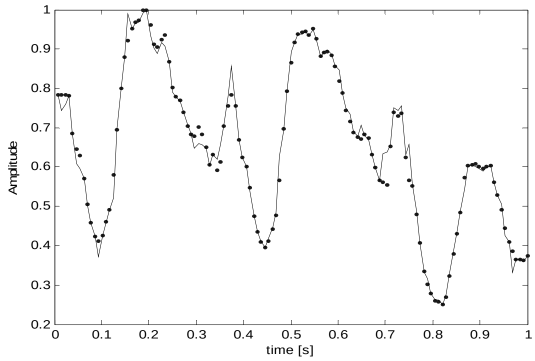

Fig. 10. Reconstructed signal using MCG (solid line) and original uniformly sampled signal (dots).

Numerical results are presented in Table 2. The table columns show the reconstruction method used, the elapsed time required for signal reconstruction and the square roots of the energies of the difference between the original and the reconstructed signal, respectively.

TABLE 2. NUMERICAL RESULTS OBTAINED IN THE SIMULATION PROCESS FOR DIFFERENT RECONSTRUCTION ALGORITHMS

Algorithm	***Elapsed time***	$\|\|x - x_1\|\|$
L_1, db2	0,296 sec.	0,3274
MCG Db2, 2 level	0,297 sec.	0,2489

IV. CONCLUSION

The algorithm proposed in this paper solves over-determined systems of linear of equations arising from non-uniformly sampled signals using L_1 – norm approximation. In contrast the MCG algorithm uses L_2 – norm approximation.

Our experiments indicate that L_1 approximation works better than MCG when there is are more samples missing. The execution efficiency (elapsed time) does not differ significantly between the two methods.

V. REFERENCES

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