Recursive Formula for the Moments of Queue Length in the M/M/c and Erlangian Queues

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***Abstract* — Problems involving moments of random variables arise naturally in many areas of science. The aim of this paper is to derive the recursive formulas of the moments of queue length distributions for the M/M/c and Erlangian queues. An application of the higher moments of queue length distribution is related to optimization problem. Starting from the basic state transition equations of the queue, we get the recursive formula for the moments of queue length distribution. This method provides an alternative approach to derive the higher moments of queue length, instead of taking the derivatives of Moment Generating Function. The presented results are continuation of previous research for M/M/1 and M/M/1/B queue presented in literature.**

***Keywords* – queue, moments of queue length, M/M/c, Erlangian distribution, optimization**

I. Introduction

THE higher moments of queue length distribution are important for optimization problem solving. Using higher moments is relevant to construct some performance bounds or to analyze the system transient behavior [1]. The problem of deriving bounds on the probability that a certain random variable belongs in a set could be solved by using given information on the moments of random variable [2]. Moments of the job size distribution, for instance, can provide bounds on the mean waiting time. However, approximations based on the first two moments of the job size distribution can not be accurate for all types of distributions. It is proved in literature that the third moment and other higher moments of the job size distribution have a significant impact on the mean waiting time [3].

Usually, the moments of queue length distribution are computed from its Moment Generating Function (MGF). In this paper, starting from the basic state transition equations of the queue, we get the recursive formulas for the moments of queue length for the different queues. This method for derivation of the recursive formulas for M/M/1 and M/M/1/B queues is presented in [1]. It is obvious that this derivation method can be applied to various queuing systems. Within the framework of this paper we will derive the recursive formulas for M/M/c, M/M/c/0, and M/E_r/1 queues, where E_r means the *r*-stage Erlangian server. The recursive formulas for the M/M/∞ and E_r/M/1 queues will be also represented.

A. Moment recursive formula for M/M/1 queue

In a M/M/1 queue with mean arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu < 1$, the *k*th moment M_k $(k \geq 1)$ of queue length satisfies [1]:

$$M_k = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} \left[(-1)^l + \rho \right] M_{k+1-l} - (-1)^{k+1} \left[1 - \rho \right]}{(k+1)(1-\rho)}, \quad (1)$$

where

$$M_0 = \sum_{n=0}^{\infty} P_n = 1 . \quad (2)$$

B. Moment recursive formula in M/M/1/B queue

In a M/M/1/B queue with mean arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu < 1$, the *k*th moment M_k^* $(k \geq 1)$ of queue length satisfies [1]:

$$M_k^* = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} \left[(-1)^l + \rho \right] M^*_{k+1-l} - (-1)^{k+1} \left[1 - \rho \right]}{(k+1)(1-\rho)}, \\ - \frac{\rho P_B \left[(B+1)^{k+1} - B^{k+1} + (-1)^{k+1} \right]}{(k+1)(1-\rho)} \quad (3)$$

where

$$M_0^* = \sum_{n=0}^{B} P_n = 1 . \quad (4)$$

The moment recursive formulas of the queue length distribution for the M/M/c, M/M/c/0 and M/M/∞ queues will be presented in Section II and Section III, respectively. Brief review of Erlangian distribution and derivation of recursive formulas for the M/E_r/1 queue length distribution and recursive formula for the E_r/M/1 queue length distribution will be presented in Section IV. Finally, in Section V we will give some conclusions.

II. Moment Recursive Formula in M/M/c queue

A. Brief review of M/M/c queue

M/M/c is the well-known Kendall's notation of a queue with *c* identical servers, where packets arrive according to a Poisson process with mean arrival rate λ and

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independent exponentially distributed service times with mean time $1/\mu$ [4]. When number of packets in system $n < c$, packet goes directly in service at the next available server, else $n \geq c$, packet is queued [5]. The offered load to the queue is $\rho = \lambda/\mu$ and the long-run server utilization is defined by $\rho = \lambda/c\mu$. Define P_n (n = 0, 1, 2, ...) as equilibrium probabilities that the queue contains n pakets (including the c in service). These probabilities satisfy corresponding equilibrium state transition equation as:

$$\begin{cases} \mu P_1 = \lambda P_0 \\ (n+1)\mu P_{n+1} = (\lambda + n\mu)P_n - \lambda P_{n-1},\ 1 \le n \le c-1 \\ c\mu P_{n+1} = (\lambda + c\mu)P_n - \lambda P_{n-1},\ n \ge c \end{cases} . \quad (5)$$

Based on (5) it is well known that we can get packet number distribution as follows:

$$P_n = \begin{cases} \dfrac{(c\rho)^n}{n!}P_0,\ n = 0,1,...,c-1 \\ \dfrac{c^c\rho^n}{c!}P_0,\ \ n = c, c+1,... \end{cases} . \quad (6)$$

Where P_0 is the is the probability of system being emty.

The average number of packets in the queue could be written as:

$$\bar{N} = M_1 = \sum_{n=1}^{\infty} nP_n = c\rho + \rho\frac{(c\rho)^c}{c!}\frac{P_0}{(1-\rho)^2}. \quad (7)$$

B. Moment recursive formula in M/M/c queue

Theorem 1: In an M/M/c queue with mean arrival rate λ and mean service time $1/\mu$, when the offered load per server $\rho = \lambda/c\mu < 1$, then the kth moment M_k ($k \ge 1$) of queue length satisfies

$$M_k = \frac{\sum_{l=2}^{k+1}\binom{k+1}{l}\left[(-1)^l + \rho\right]M_{k+1-l} - (-1)^{k+1}P_0}{(1-\rho)(k+1)} - \frac{\sum_{n=1}^{c-1} n^{k+1}P_0\frac{(\rho c)^n}{n!}\left(\rho\left(\frac{c}{n+1}-1\right) - \left(1-\frac{n}{c}\right)\right)}{(1-\rho)(k+1)}. \quad (8)$$

Proof:

When $k \ge 1$, define $M_k = \sum_{n=0}^{\infty} n^k P_n$. Multiplying n^{k+1} on both sides of the third equation of (5) yields:

$$c\mu n^{k+1}P_{n+1} = (\lambda + c\mu)n^{k+1}P_n - \lambda n^{k+1}P_{n-1}. \quad (9)$$

We can further write (9) as:

$$\begin{aligned} & c\mu\left[(n+1)-1\right]^{k+1}P_{n+1} = \\ & = (\lambda + c\mu)n^{k+1}P_n - \lambda\left[(n-1)+1\right]^{k+1}P_{n-1} \end{aligned} \quad (10)$$

$$\begin{aligned} & c\mu\left[\sum_{l=0}^{k+1}\binom{k+1}{l}(n+1)^{k+1-l}(-1)^l\right]P_{n+1} \\ & = (\lambda + c\mu)n^{k+1}P_n - \lambda\left[\sum_{l=0}^{k+1}\binom{k+1}{l}(n-1)^{k+1-l}\right]P_{n-1} \end{aligned} . \quad (11)$$

Now let n = c, ..., ∞, respectively in (11) and sum up all these equations together, we obtain:

$$\begin{aligned} & c\mu\sum_{l=0}^{k+1}\binom{k+1}{l}(-1)^l\left[\sum_{n=c}^{\infty}(n+1)^{k+1-l}P_{n+1}\right] \\ & = (\lambda + c\mu)\sum_{n=c}^{\infty}n^{k+1}P_n - \lambda\sum_{l=0}^{k+1}\binom{k+1}{l}\left[\sum_{n=c}^{\infty}(n-1)^{k+1-l}P_{n-1}\right] \end{aligned} . \quad (12)$$

$$\begin{aligned} & \sum_{l=2}^{k+1}\binom{k+1}{l}\left[(-1)^l + \rho\right]M_{k+1-l} - (-1)^{k+1}P_0 - \sum_{n=1}^{c-1}\frac{(\rho c)^{n+1}}{(n+1)!}n^{k+1}P_0 = \\ & = (1-\rho)(k+1)M_k - (1+\rho)\sum_{n=1}^{c-1}\frac{(\rho c)^n}{n!}n^{k+1}P_0 + \rho\sum_{n=1}^{c-1}\frac{(\rho c)^{n-1}}{(n-1)!}n^{k+1}P_0 \end{aligned} \quad (13)$$

Based on the previous equation we could obtain the formula (8). For c = 1 in the (8), we get the recursive formula for M/M/1 queue [1].

III. Moment Recursive Formula in M/M/c/0 and M/M/∞ Queues

A. Moment Recursive Formula in M/M/c/0 queue

The M/M/c/0 queue is a variation of the M/M/c queue with the waiting space removed. A packet either is admitted to service or, if all the servers are occupied, denied admittance and is dropped from the system [6]. The offered load to the queue is $\rho = \lambda/\mu$. The equilibrium state transition equations could be written as:

$$\begin{cases} \mu P_1 = \lambda P_0 \\ (n+1)\mu P_{n+1} = (\lambda + n\mu)P_n - \lambda P_{n-1},\ 1 \le n \le c-1 \\ c\mu P_n = \lambda P_{n-1},\ n = c \end{cases} . \quad (14)$$

P_c is the packet blocking probability given as follows:

$$P_c = \frac{\rho^c}{c!\sum_{k=0}^{c}\frac{\rho^k}{k!}}. \quad (15)$$

Based on (14) and (15), we can obtain packet number distribution as:

$$P_n = \frac{\rho^n}{n!}P_0, \text{ where } n = 1, 2, 3\ldots \quad (16)$$

The average number of packets could be written as:

$$\bar{N} = M_1 = \rho\left(1 - \frac{\rho^c}{c!\sum_{k=0}^{c}\frac{\rho^k}{k!}}\right). \quad (17)$$

Theorem 2: In an M/M/c/0 queue with mean arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu < 1$ and $M_0 = \sum_{n=0}^{c} P_n = 1$, then the kth moment M_k ($k \ge 1$) of queue length satisfies:

$$M_{k+1} = \frac{\sum_{l=2}^{k+1}\binom{k+1}{l}\left[(-1)^l \mu M_{k+2-l} + \lambda M_{k+1-l}\right]}{\mu(k+1)} + \frac{\lambda(k+1)M_k + \lambda P_C(C^{k+1} - (C+1)^{k+1})}{\mu(k+1)} \quad (18)$$

Proof:

When $k \geq 1$, define $M_k = \sum_{n=0}^{c} n^k P_n$. Multiplying n^{k+1} on both sides of the second equation of (14), using the same method as before yields:

$$(n+1)\mu n^{k+1} P_{n+1} = (\lambda + n\mu) n^{k+1} P_n - \lambda n^{k+1} P_{n-1} \quad (19)$$

$$\begin{aligned}&(n+1)\mu\left[\sum_{l=0}^{k+1}\binom{k+1}{l}(n+1)^{k+1-l}(-1)^l\right]P_{n+1} \\ &= (\lambda + n\mu)n^{k+1}P_n - \lambda\left[\sum_{l=0}^{k+1}\binom{k+1}{l}(n-1)^{k+1-l}\right]P_{n-1}\end{aligned} \quad (20)$$

$$\begin{aligned}&\mu\sum_{l=0}^{k+1}\binom{k+1}{l}(-1)^l\left[\sum_{n=1}^{c-1}(n+1)^{k+2-l}P_{n+1}\right] \\ &= \sum_{n=1}^{c-1}(\lambda + n\mu)n^{k+1}P_n - \lambda\sum_{l=0}^{k+1}\binom{k+1}{l}\left[\sum_{n=1}^{c-1}(n-1)^{k+1-l}P_{n-1}\right].\end{aligned} \quad (21)$$

Based on the previous equation we get the formula (18).

B. Moment Recursive Formula in M/M/∞ queue

The case of M/M/∞ queue may be interpreted either as that of a responsive server who accelerates its service rate linearly when more customers are waiting or may be interpreted as the case where there is always a new server available for each arriving customer [7]. Here, we consider that M/M/∞ queue has infinite number of servers [8]. If mean arrival rate of the queue is λ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu <$ 1, and $M_0 = \sum_{n=0}^{\infty} P_n = 1$, then the kth moment M_k $(k \geq 1)$ of queue length satisfies:

$$M_{k+1} = \frac{\sum_{l=2}^{k+1}\binom{k+1}{l}\left[(-1)^l \mu M_{k+2-l} + \lambda M_{k+1-l}\right] + \lambda(k+1)M_k}{\mu(k+1)}. \quad (22)$$

Due tu lack of space we didn't demonstrate complete proof for the previous formula.When c in (8) increases to ∞, we get the previous equation. Starting from the equilibrium state transition equations [7] and fallowing the same method as above we could derive the proof for the recursive formula of M/M/∞ queue . Actually, from (22) we get the second moment as:

$$M_2 = \rho(\rho+1) \quad (23)$$

For the purpose of verification of the recursive formula we could compare the result obtained using classic way:

$$\begin{cases} P(z) = e^{\rho(z-1)} \\ M_2 = \sum_{i=0}^{\infty} i^2 \frac{\rho^i}{i!} e^{-\rho} = \left.\frac{d^2P(z)}{dz^2}\right|_{z=1} + \left.\frac{dP(z)}{dz}\right|_{z=1} = \rho(\rho+1) \end{cases}. \quad (24)$$

IV. Erlangian Distribution

The "method of stages" permits one to study queuing systems that are more general than the birth-death systems. Erlang recognized the simplicity of the exponential distribution and its great power in solving Markovian queuing systems. The key principle of the method of stages is the memoryless property of the exponential distribution. This lack of memory is reflected by the fact that the distribution of the time remaining for an exponentially distributed random variable is independent of the acquired "age" of that random variable. The state of the service facility may be described by merely giving the number of stages yet to be completed by a customer in service [7].

A. Moment recursive formula in M/ E_r /1 queue

Besides specifying the number of customers in the system, the number of stages remaining in the service facility for the customer in service must be also specified. Each customer in the queue is represented as possessing r stages of service yet to be completed for him. Thus the state variable is taken as the total number of service stages yet to be completed by all customers in the system at the time the state is described. In particular, if we consider the state at the time when the system contains k customers and when the ith stage of service contains the customer in service then the number of stages contained in the total system is [7]:

j = number of stages left in total system = $(k-1)r + (r-i+1) = rk - i + 1$.

For this case p_k is defined as the equilibrium probability for the number of customer in the system. We further define:

$P_j = P$ (j stages in system).

The relationship between number of customers and stages allows us to write

$$p_k = \sum_{j=(k-1)r+1}^{kr} P_j, \; k = 1,2,3... \quad (25)$$

Focusing on stage n we see that transition to stage n from a state which is r positions below (stage n-r) is due to the arrival of r new stages when a new customer enters. Transition to stage n from the state n+1 is due to the completion of one stage within the r-stage service facility. Furthermore, state n may be left at the rate λ due to an arrival and at a rate $r\mu$ due to a service completion [5]. In this system the mean arrival rate is λ and the average service time is held fixed at $1/\mu$ independent of r. Thus the utilization factor is $\rho = \lambda/\mu$. The equilibrium state transition equations can be written by using flow conservation method:

$$\begin{cases} \lambda P_0 = r\mu P_1 \\ r\mu P_{n+1} = (\lambda + r\mu)P_n - \lambda P_{n-r},\ 1 \le n \le \infty \end{cases}. \quad (26)$$

Theorem 3: In an M/ E_r /1 queue with mean arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu <$ 1, and $M_0 = \sum_{n=0}^{\infty} P_n = 1$, the *k*th moment M_k $(k \ge 1)$ of queue length satisfies:

$$M_k = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} \left[(-1)^l r + r^l \rho\right] M_{k+1-l} - r(-1)^{k+1}(1-\rho)}{r(k+1)(1-\rho)} \quad (27)$$

Proof:

When $k \ge 1$, define $M_k = \sum_{n=0}^{\infty} n^k P_n$. Multiplying n^{k+1} on both sides of the equation of the second equation of (26) yields:

$$r\mu n^{k+1} P_{n+1} = (\lambda + r\mu) n^{k+1} P_n - \lambda n^{k+1} P_{n-r} \quad (28)$$

$$r\mu \left[\sum_{l=0}^{k+1} \binom{k+1}{l} (n+1)^{k+1-l} (-1)^l \right] P_{n+1} = (\lambda + r\mu) n^{k+1} P_n - \lambda \left[\sum_{l=0}^{k+1} \binom{k+1}{l} r^l (n-r)^{k+1-l} \right] P_{n-r} \quad (29)$$

Now let n = 1, ..., ∞, respectively in equation (29) and sum up all these equations together, we obtain:

$$r\mu \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^l \left[\sum_{n=1}^{\infty} (n+1)^{k+1-l} P_{n+1} \right] = (\lambda + r\mu) \sum_{n=1}^{\infty} n^{k+1} P_n - \lambda \sum_{l=0}^{k+1} \binom{k+1}{l} r^l \left[\sum_{n=1}^{\infty} (n-r)^{k+1-l} P_{n-r} \right] \quad (30)$$

$$r\mu \sum_{l=2}^{k+1} \binom{k+1}{l} (-1)^l M_{k+1-l} - \mu r (-1)^{k+1} P_0 + r\mu M_{k+1} - (k+1) r\mu M_k = (\lambda + r\mu) M_{k+1} - \lambda \sum_{l=2}^{k+1} \binom{k+1}{l} r^l M_{k+1-l} - \lambda M_{k+1} - \lambda r (k+1) M_k \quad (31)$$

Based on the previous equation we could obtain the formula (27). When r = 1 in the previous equation, we get the recursive formula for M/M/1 queue (1).

B. *Moment recursive formula in E_r /M/1 queue*

Here the roles of interarrival time and service time are interchanged from those of the previous section; in many ways these two systems are duals of each other [7]. Using the same method as above for E_r /M/1 queue with mean arrival rate $r\lambda$ and mean service time $1/\mu$, when the offered load $\rho = \lambda/\mu$ < 1, and $M_0 = \sum_{n=0}^{\infty} P_n = 1$, we get the recursive formula for the *k*th moment M_k $(k \ge 1)$ of queue length as follows:

$$M_k = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} \left[(-r)^l + r\rho\right] M_{k+1-l} + (-r)^{k+1} P_0}{r(k+1)(1-\rho)} - \frac{\sum_{n=1}^{r-1} n^{k+1} P_0 \left(\frac{(\rho r)^{n+r}}{(n+r)!} + (1+r\rho)\frac{(\rho r)^n}{n!} - r\rho \frac{(\rho r)^{n-1}}{(n-1)!} \right)}{r(k+1)(1-\rho)}. \quad (32)$$

When r = 1 in the previous equation, we get the recursive formula for M/M/1 queue (1). Due tu lack of space we didn’t demonstrate complete proof for the previous formula. The proof could be derived by using described method starting from the basic equilibrium state transition equations presented in literature [7].

V. CONCLUSION

The moment recursive formulas of the queue length distribution for the M/M/c and Erlangian queues is presented. Presented method is based on previous research for M/M/1 and M/M/1/B queue proposed in literature. Recursive relationship between the moments of queue length is derived using equilibrium state transition equations. Higher moments of queue length distribution are applied in solving of optimization problem.

REFERENCES

[1] J. Liu., H. Jiang, and H. Susumu, “Recursive formula for the moments of queue length in the M/M/1 queue”, *IEEE communications letters*, vol. 12, No. 9 , September 2008

[2] D. Bertsimas, I. Popescu and J. Sethuraman, “Moment Problems and Semidefinite Optimization”, *Sloan School of Management and Operations Research Center, Cambridge*, April 2000

[3] V. Gupta, J.Dai, M. Harchol-Balter and B. Zwart, “The effect of higher moments of job size distribution on the performance of an *M=G=K* queueing system”, *School of Computer Science, Carnegie Mellon University, Pittsburgh, PA, USA*, February 2008

[4] J. R. Artalejo and M. Pozo, “Numerical Calculation of the Stationary Distribution of the Main Multiserver Retrial Queue”, *Springer*, vol. 116, pp 41-56, October 2002

[5] I. Adan and J. Resing, “Queueing Theory”, *Eidhoven University of Tehnology*, February 2001

[6] Telcom 2120, “Additional Markovian Queues”, *University of Pittsburgh*, 2004

[7] L. Kleinrock., “Queueing systems”, *John Wiley & Sons*, New York (1976)

[8] Y. A. Korilis, “Networking Theory & Fundamentals”, *University of Pennsylvania*, Spring 2003